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On Amsler's Planimeter. *By* F. J. Bramwell, *C.E.*

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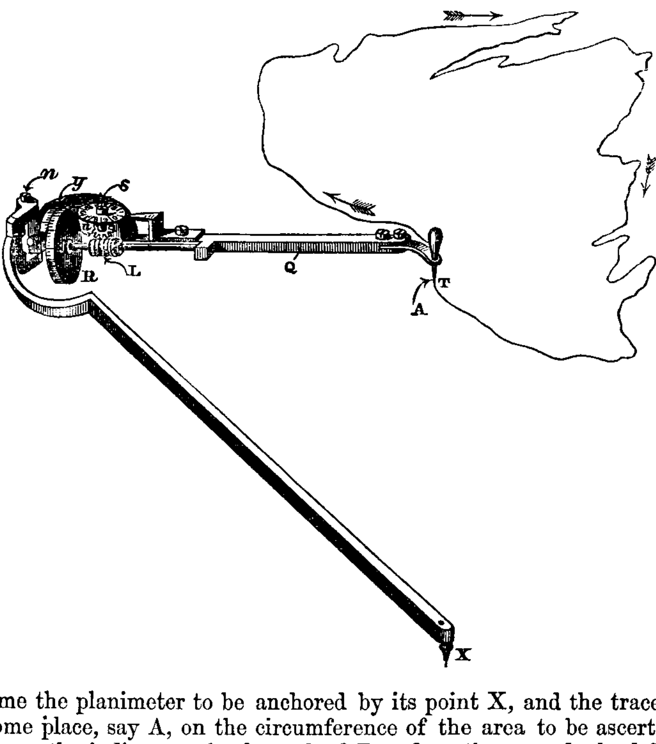
This machine for measuring the area of any figure, however irregular, by the mere passage of a tracer round about its perimeter, has now been in use for some years; but, so far as the writer is aware, no easily intelligible statement of its principles of action has ever been made public.

Although no doubt the mere construction of the planimeter is now generally known, it may enable the explanation which is about to be offered to be more easily followed if a sketch of the actual machine, as at work upon a map, be given here (see fig. 1).

Fig. 1.



Assume the planimeter to be anchored by its point X, and the tracer T to be at some place, say A, on the circumference of the area to be ascertained; and assume the indices on the first wheel R and on the second wheel S to be at zero, and that then the tracer T be carried along the perimeter of the area in the direction of the arrows (with the sun), the indices will give a reading up to four figures, which will represent square inches, to two places of whole numbers and to two places of decimals.

This movement of the indices is effected by the wheel R, the edge of which bears upon the paper, so that as the tracer T is made to go round about the figure to be measured, the wheel R, from its contact with the paper, receives rotary motion, and by means of the worm-pinion L and worm-wheel *u*, communicates a diminished motion (1-10th) to the horizontal wheel *s*.

The circumference of the wheel R is "divided," and it works against a vernier at y; the horizontal wheel s gives "tens" in square inches, the larger divisions on the travelling wheel R "units," the smaller divisions on that wheel "tenths," and the vernier "hundredths" of square inches. All that has to be done for ascertaining an area is to read the indices after the machine is anchored and the tracer is put to the starting-point; but before it is started, to book the reading, to re-read after the circuit of the figure has been made, and then to deduct the first reading from the second; the remainder gives the area (in square inches and decimals) of the particular figure.

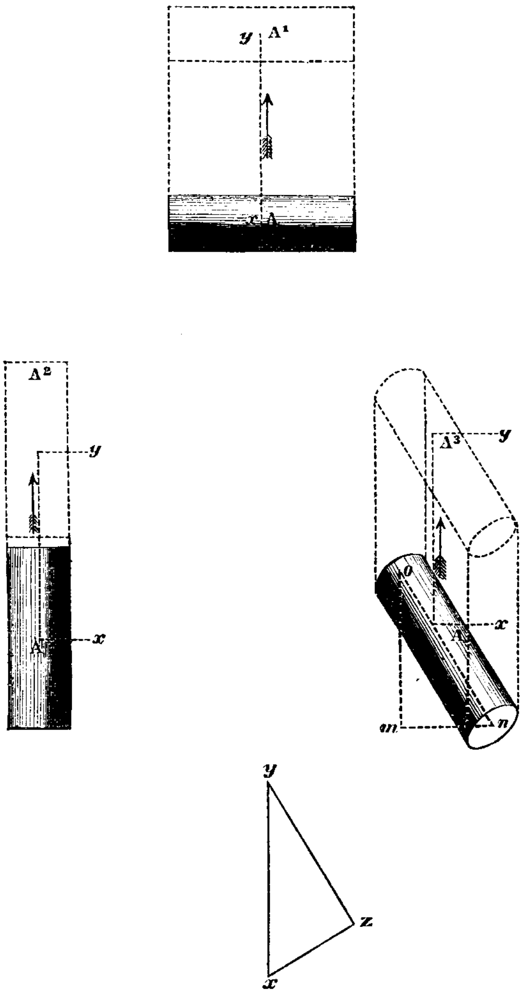
The foregoing being, briefly stated, the construction, the manner of using, and the result of that using of the planimeter, it now remains to endeavour to show, as intelligibly as possible, why it is that such an implement, by merely following the boundary of a figure, should give with absolute accuracy the area of that figure.

Such a proposition at first sight appears to involve an impossibility. One is in the habit of saying, and of most truly saying, that there is no fixed relation between perimeter and area; and of saying, moreover (and also truly), that not only is this the fact when areas of great irregularity are dealt with, but, as regards direct proportion, it is also the fact when the most regular figures (figures in all respects the same, except in their actual size) are under consideration; for it is as true that the circumferences of perfectly regular figures like circles bear no more fixed direct proportion to the areas of those circles, unless the exact size be known, as it is true that the coast-line of Norway, indented with its deep fjords, bears no more relation to the area of that romantic country than the perimeter of a prosaic rectangular portion of the United States bears to the square miles of prairie contained within it. These things being so, it does, as has already been said, seem at first sight absurd to endeavour to obtain from the traverse of a perimeter, be that perimeter the most regular imaginable (and if possible still more absurd when that perimeter may be the most irregular imaginable), the correct area contained within it, not merely in terms of the perimeter, but in a definite standard measurement, such as square inches.

As a preliminary to the investigation of the action of an elementary planimeter, let the results of the moving of a plain cylinder in contact with a flat surface, and under certain varying conditions, be considered.

Assume a cylinder, as A in fig. 2, and that it is intended to move that cylinder parallel with itself in the direction shown by the arrow, over the length $x\ y$. The cylinder may be (1st) at right angles to the direction in which it is to be traversed, as in A A^1. If under these circumstances the cylinder be moved from x to y and brought into the position as dotted at A^1, the motion will be entirely one of rolling, without any sliding whatever; and if there were upon the surface a trace ($x\ y$) of ink capable of making a mark upon the cylinder, there would be found circumferentially upon it, when it had reached the new position, a line, the length of which would be equal to $x\ y$. (2nd) The cylinder may be placed with its axis parallel to the direction of motion, as at A A^2; then no rolling action would take place, but the cylinder would simply slide endways upon the surface. The cylinder would, however, still bear upon it the trace $x\ y$, equal in length to the distance it had moved through, but that trace would be obviously a mere straight line in the direction of the axis of the cylinder. (3rd) The cylinder may be in a position intermediate between that of A A^1 and A A^2; that is to say, may be neither at right angles to the line of motion, as in A A^1, nor parallel with the line of motion, as in A A^2, but at an angle therewith, as in A A^3. In this instance, on the cylinder being caused to traverse from x to y, the motion will be one compounded of rolling and of sliding; the trace will still be made on the cylinder; the length of that trace will be, as before, the length $x\,y$, but the trace will now be a spiral, which may be developed into the triangle $x\,y\,z$, and the base $x\,z$ will bear such a relation to the hypotenuse $x\,y$ as the base $m\,n$ of the triangle $m\,n\,o$ bears to the hypotenuse $n\,o$. But it has been

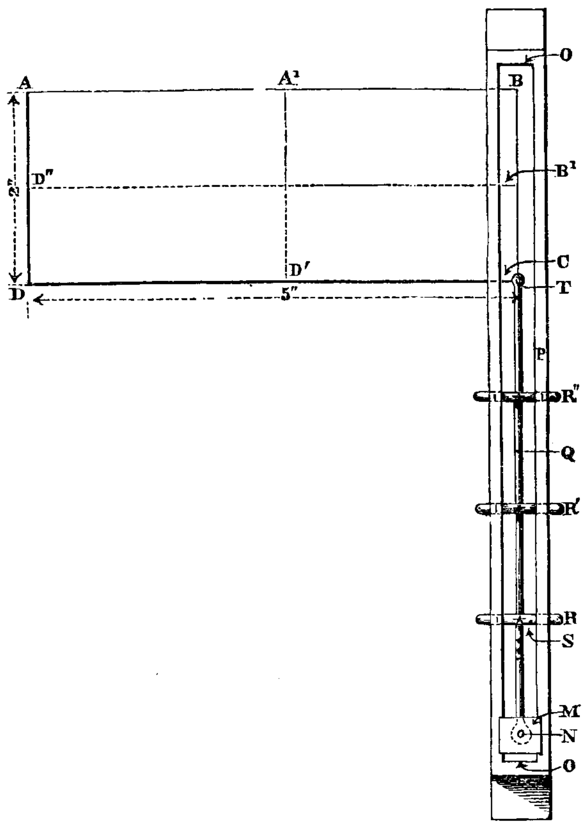
Fig. 2.



said that in the journey from x to y the cylinder will have had a motion compounded of sliding and of rolling; the extent of the rolling will clearly bear that proportion to the total traverse $x\,y$ that the base $m\,n$ bears to the hypotenuse $n\,o$; and this proportion may obviously be any thing between the absolute equality which would exist in A A^1 down to the absence of all rolling motion which would obtain in the case of A A^2.

These preliminaries being stated, let it be inquired how they apply to the action of the planimeter. For this purpose it will be well to refer to the sketch, fig. 3. This sketch shows an imaginary elementary planimeter, used

Fig. 3.



to ascertain the area of the rectangle A B C D, the length of each of its sides A B, C D being 5 inches, and the length of each of its ends D A, C B being 2 inches, so that its area is 10 inches. Let M be a block carrying the pivot N and capable of sliding in the straight groove O O in the bridge P, pinned down over the paper, and let Q be a rod pivoted at N, and say, for the sake of illustration, 5 inches long from the pivot N to the tracer T at its opposite end; and let it have on it, say at R, a wheel R, having a circumference of exactly 2 inches; and also, for the sake of a second illustration, let there be similar wheels as R′, R″ free to revolve on the rod Q, at distances greater than the distance of the wheel R from the pivot N; and let there be to one of the wheels, say R, a pointer S, to enable the graduated divisions on the circumference of R to be read off.

Now let it be assumed that the tracer T is moved from C to D; the result will be that during the motion the block M will gradually pass along the groove O until the time when the tracer T has reached D; and then, as the length of the rod Q is exactly 5 inches, equal to the length of the side C D (5 inches), the block M must have passed along the groove O until the centre N in that block is immediately over the point C, and the centre line of Q is coincident with the line C D. If, now, the tracer T be moved along the 2 inches from D to A, the block M must move parallel with it, and the axis Q of the wheels R, R′, R″ will therefore be at right angles to the line of motion, and the wheels themselves will, like the cylinder A in A A^1 of fig. 2, have a rolling motion, and a rolling motion only; and thus by the time the tracer T has reached the point A, these wheels will each have made an entire revolution. If, now, the circumference of R or R′, R″ has been divided into ten equal parts, and if on setting out from D pains had been taken to put the wheel R with its zero mark to the pointer S, it would be found, on the arrival of the wheel at A, that it had made an entire revolution, and that therefore the index would read 10, equal 10 square inches—viz. the multiplication of the length of the radius Q (5 inches) into the circumference of the wheel R (2 inches).

Now let it be assumed that the implement is to be used for the purpose of measuring another rectangle A B C D, also of 10 inches area, having its sides and ends respectively 2 inches and 5 inches long; so that in this instance (see fig. 4) the ends have the 5-inch measurement in lieu of the 2, and the sides have the 2-inch in lieu of the 5. Once more let the tracer T be moved from C to D; the block M will now have only passed along the groove O a comparatively insignificant distance towards C, and the rod Q will lie at the angle shown, so that it will form the hypotenuse (5 inches long) of a triangle of which the base will be C D (2 inches long). If, now, the tracer T be moved from D to A (5 inches), the block M will make a similar motion in the groove O; and when the tracer T has reached A, the rod Q will have moved parallel to itself, and will be found in the position shown in fig. 5. But, as has already been said when speaking of A A^3 of fig. 2, if a cylinder capable of rotating be caused to move over and in contact with a surface when it is in a position neither parallel with, nor at right angles to, the line of motion, and if it be made to preserve its own parallelism, the result will be a motion compounded of sliding and of rolling, and the amount of the rolling will bear such a relation to the whole motion as the base *m n* bears to the hypotenuse *n o*. In the instance, therefore, under consideration the ratio of revolution to the whole motion will be that of 2 to 5; therefore if the zero on the wheel R were brought to the pointer S at the time of setting out from D, it would be found, when the tracer had arrived at the end A of its 5-inch journey D A, that the wheel R would have made just one revolution, and that the figure 10, indicating 10 square inches, would present itself.

From a consideration of the foregoing two cases, it will be seen that the "rate" of rotation of the wheel R, when it moves along the line D A, depends upon the length of the line C D, and the "quantity" of such rotation upon that of the line D A. These two expressions, "rate" and "quantity," will be used hereafter in the above senses.

As an illustration of "rate" and "quantity," suppose that the rectangle of fig. 3 had only been half as long as the one that has been considered, namely $2\frac{1}{2}$ inches, and had been bounded by the line D′ A^1; if, then, the tracer had been moved from D′ to A^1, the "rate" of revolution of the wheels R &c. would have been one half of the total distance moved through by the tracer, because C D′ (equal $2\frac{1}{2}$ inches) is one half of the length of the rod Q. The "quantity" of motion in going along D′ to A¹ would, however, have been the same as it was in passing from D to A, because D′ A¹ equals D A; but an

Fig. 4.

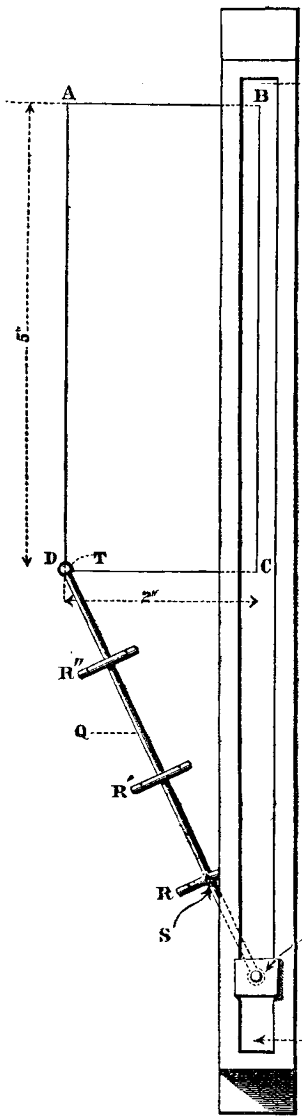
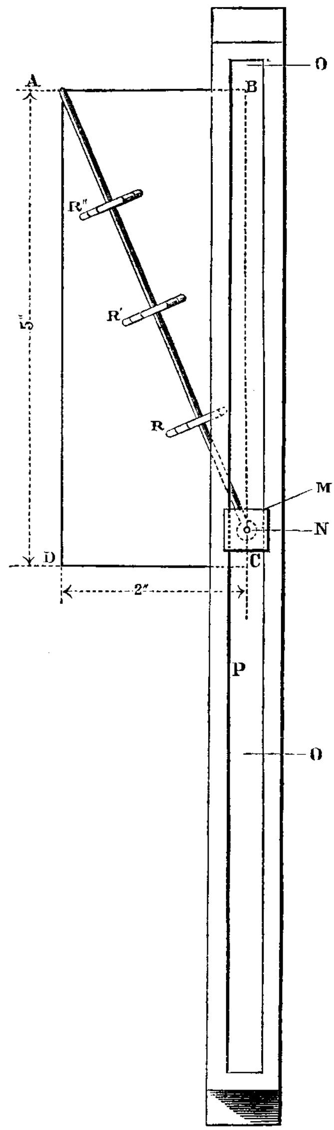


Fig. 5.



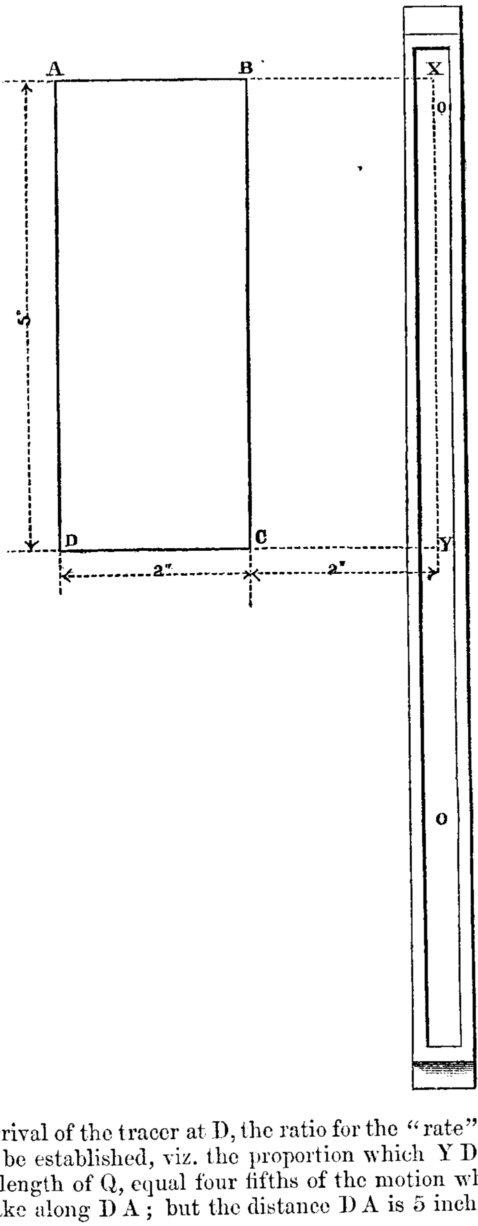
equal "quantity" into half the "rate" will only give half the total amount, and therefore the wheels R would have recorded a half revolution, equal 5 square inches, thus accurately giving the area C D′, A¹ B. On the other hand, assume that the height of the rectangle had been halved, and that it had been bounded by the lines C D, D'' B¹, then the wheels R &c. in traversing from D to D'' would do so at their full "rate" of revolution, the line C D being 5 inches long; but the "quantity" of such revolution would only be half that which it was in going from D to A, because D D'' is only half D A, and therefore the wheels again would register but a half revolution, indicating truly the 5-inch area of the 5-inch by 1-inch parallelogram D D'', B¹ C.

In each of the foregoing cases it has been assumed that the index is read when the apparatus is about to start from D, and is re-read when it reaches A. Such a reading would be quite sufficient in the case of a rectangle where the groove O O is assumed to be in the prolongation of one of the sides (B C); but under any other circumstances the complete circuit of the figure must be made. To test this, let it be assumed that the tracer T starts from C, and that the index on R is read just before the starting, and then let it be examined when the tracer T has reached D; it will be found that the wheel R has received an amount of rotation approximately that due to its traversing the arc of the radius N R, that R' has received a larger amount of traverse, and R'' a still larger amount, owing to their greater distance from the centre N; but it will be afterwards found that these amounts of revolution may be wholly neglected, and that they will not come into the final computation, because, assume the tracer T to have attained to the point A and to have put into the wheels R, R', R'' the one revolution which it has been seen that traverse would give, those wheels would be found at A (were there any means by multiplying gear, as in the actual machine, to record more than the one revolution) to have made the one revolution each, plus the varying amounts of revolution which they would have received in their journey from C to D. But in their back journey from A to B it is manifest they will each of them unwind (if such a phrase may be used) exactly the quantity of revolution which was put into them in moving from C to D. Further, during the passage from B to C to complete the circuit, the direction of motion being parallel with the position of the rod Q, the axle of the wheels R, R', R'', no rolling movement will be communicated to them, as they will be in the condition of the cylinder A A² of fig. 2, and will merely slide over the paper, so that on the arrival of the tracer T at C, having made the circuit of the rectangle, there will be found in them the one revolution, and neither more nor less than the one revolution, generated by the traverse from D to A.

The next point to be proved is the manner in which the implement will truly record if the groove O O be not on the line produced by prolonging one side of the rectangle. Let fig. 6 represent a rectangle, say 2 inches long on its side C D and 5 inches high at its end D A, and containing therefore 10 square inches, and let X Y be a line parallel with B C, and as far removed (2 inches) on the right hand from it as D A is removed from it on the left hand, and let the groove O O be on the line X Y; then, if the tracer T were to stand at C, and the wheels R &c. were at zero, and if the tracer were then moved along the line C B, there would be put an amount of revolution into R which would be compounded of the "rate" due to the length Y C and of the "quantity" belonging to the length C B, or 2 multiplied by 5 equal 10 inches, equal one revolution of R. But if now the tracer T be brought back again along the line B C, the wheel R will unwind the revolution that was put into it, and on its return to C will be found at zero.

Having thus premised that during the passage of the tracer T from B to C the wheel R will have unwound or made a negative quantity expressive of the rectangle B X Y C, let the measurement of A B C D be considered. Assume the tracer to start from C, and the wheels R &c. to be at zero, then in the passage from C to D varying revolutions would be put into these wheels corresponding approximately with the length of their arcs about the centre N;

Fig. 6.



then, on the arrival of the tracer at D, the ratio for the "rate" of trace between D and A will be established, viz. the proportion which Y D (4 inches) bears to the 5-inch length of Q, equal four fifths of the motion which the tracer T is about to make along D A; but the distance D A is 5 inches, and therefore the wheels R &c. will make a further 4 inches of circumferential movement, equal 2 revolutions, indicating 20 square inches. If, now, the tracer T be moved from A to B, there will clearly be unwound from all the wheels R &c. the amount of motion that was put into them in traversing from C to D, and thus the wheels R &c. will all be left with the double revolution indicative of 20 square inches. The only side remaining to be passed over is that from B to C; and if this traverse were devoid of effect on the wheels R &c., as the traverse from B to C was in the cases of figures 3, 4, and 5, then the implement on arriving at C, at the end of the circuit, would record double the proper area, or 20 inches instead of 10; but in the outset of this paragraph it was shown that the journey from B to C in fig. 6 would unwind exactly one revolution of the wheel R, leaving therefore one revolution remaining, indicating, as it should do, 10 square inches for the area of A B C D.

The next step is to show the ability of the implement to give the area correctly of figures which are not rectangular. Assume, as in figure 7, it be

Fig. 7.

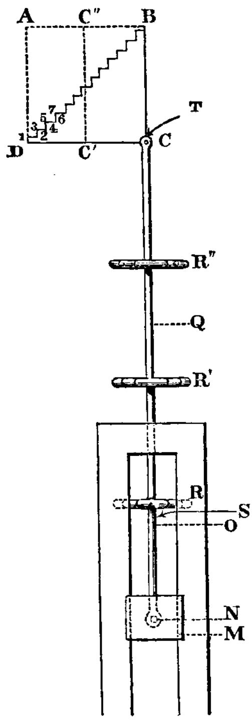
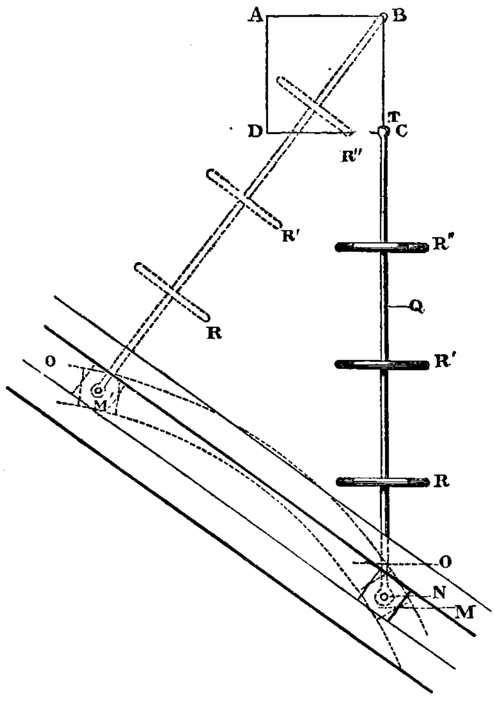


Fig. 8.



required to find the area of the triangle B C D, and let it be imagined that in lieu of the straight line for the hypotenuse B D the boundary of the figure on that side were made by a number of extremely small steps, as sketched; if then the tracer T be made once more to traverse from C to D, the wheel R will have a certain amount of revolution given to it; and if it then be made to rise through the space D 1, it will have a "rate" of revolution equal to the length of the line C D, and a "quantity" equal to the height D 1; if it then pass along the horizontal line 1 2, it will unwind that proportion of the revolution, put in on going from C to D, that is represented by the length of the line 1 2. If, now, it be made to rise from 2 to 3, it will have a "rate" of revolution equal to the length of the line C D—1 2, and a "quantity" equal to the height of the line 2 3. If it now be carried along the horizontal line 3 4, another portion of the revolution given by C D will be taken out; and then if it be made to rise from 4 to 5, a further portion of a revolution will be put in, having for its "rate" the length of the line C D—D 4, and for its "quantity" the height of the line 4 5. This may be followed through all the steps into which the hypotenuse has been broken up, and then it will be found, as is obvious, that the sum of all the horizontal lines 1 2, 3 4, 5 6, &c. is equal to the length C D, and that the traversing of them will therefore have unwound all the revolution that the passage along C D had put into the wheel R; but it will also be found that the sum of all the vertical lines 2 3, 4 5, 6 7, &c. is equal to D A; and therefore the "quantity" of revolution given to the wheel R will be equal to that which it would have had, had it passed up the line D A, while the means of the lengths of C D—1 2, C D—D 4, C D—D 6, &c. will exactly equal the half of C D, and thus the condition of the wheel R in relation to the index S will, when it arrives by the zigzag path at B, be precisely the same as it would have been if it had gone by the way of the rectangle C C′ C″ B, C C′ being half of C D. A large number of very small steps have been taken in lieu of the straight line hypotenuse D B. Obviously a greater number of much smaller steps, or an infinite number of infinitely little steps, may be substituted, until the traverse ceases to be made along steps at all, and becomes one along the slope line D B, in which condition of things the wheel R at any part of the traverse of the tracer along the hypotenuse is making a revolution compounded of the "rate" due to its horizontal distance from C, and of a "quantity" equal to the rise from D. The "quantity" remains constant during the whole journey, but the "rate" regularly diminishes, and the mean of all the "rates" is that due to the proportion that half the length of the line C D bears to N T, the length of Q.

Now if it has been proved that this elementary planimeter, no matter where anchored, can act efficiently in ascertaining the area of rectangles and of triangles, it is self-evident that it could truly ascertain the area of any other figure, because there is no figure from that of the regular circle to that of the most irregular boundary which cannot be represented by an indefinite number of straight lines lying at various angles—that is to say, a circle is only a polygon of an infinite number of sides, all equal; and any irregular figure may be divided into an indefinite number of sides, most probably unequal.

It may now be said that the elementary planimeter has been shown to have its pivot N attached to the guide-block M working up and down in the straight groove O, that that groove has been sketched with its axis either in the prolongation of B C or in a position parallel to B C, whereas in the actual planimeter there is no such straight groove at all; but the pivot N is at the end of a radius rod, which in its movement causes N to pass through the arc of a circle, and that that arc may have its chord in almost any position in relation to the line B C, and thus there are disturbing causes in the planimeter as manufactured which do not exist in the elementary planimeter. The answer to this objection, which at first sight appears so well-grounded a one, is that these differences between the real and the elementary planimeter may be left out of consideration altogether, as they really have no effect whatever upon the action of the implement. This can be made clear in a very few words.

Assume, as in fig. 8, that the groove O O were placed at an angle to the prolongation of the line B C. If, now, the tracer T be carried along the straight line from C to B, the block M will have moved along the groove O to M, and the wheel R will be found at R′; this will have communicated an amount of revolution to the wheel R due to its change of position to R; the other two wheels (R′, R″) will also have made movements depending principally on their distance from N. Such revolution of R will be given without reference to any area to be measured by the traverse of the tracer T, for that has merely passed along the straight line C B. But on bringing the tracer T back to C, the block M and wheels R, R′, R″ will be restored to the positions they held at the outset, and in being so restored the whole amount of revolution put into the wheels R &c. will be unwound.

But assume that the tracer T, instead of being carried along the line C B and back again, had been taken along the sides of the square C D A B back to C, the pivot N would return to identically the place that it had before the circuit was commenced; and whether during that circuit N moved in the groove O O as placed parallel to the prolongation of C D in fig. 3, or in it as inclined and as shown by full lines in fig. 8, or inclined and curved as dotted in that figure, could make no difference in the final result, because whatever amount of revolution might be given to the wheels R &c. by the movement of N along the path of the groove O (be that groove straight or curved, inclined or not inclined) would be taken out of them again on the return journey along that same path.

Three wheels (R, R′, R″) have been shown loose on the axle Q of the elementary planimeter; this, as was said, has been done for the mere purpose of illustration, to show that wherever situated they will register just the same.

In the actual machine as manufactured and sold, the position of the wheel is about that which has been given to R, and in this position it serves to support the hinge-joint, and is sufficiently far from the tracer T to get rid of the danger of lifting the wheel from the paper if the tracer T were held a little too high.

It is hoped it has been made clear that one revolution of the wheel R will always express an area equal to the circumference of that wheel multiplied into the length of the rod Q, the radius N T *.

If these elements are constant, the scale of the planimeter reading is constant; but if these be capable of variation, then the scale can be varied. Advantage is taken of this property in the construction of one form of the implement in which the length N T is made adjustable, and thus the instrument may be readily arranged to read either French or English superficial measure.

The purposes for which the planimeter may be applied are very numerous. It gives to the Surveyor the readiest means of calculating the acreage of whole estates or of separate fields. To the Hydraulic Engineer it affords a mode for ascertaining with ease and certainty the drainage area of a country, or the area of the sections of rivers, an important thing when it is desired to obtain the dimensions of numerous sections of a stream to ascertain its hydraulic mean depth. To the Naval Architect it presents itself as an aid in calculating the areas of the successive sections of a vessel, and thus most materially assists him in readily determining not merely the total displacement of a vessel, but those more complex problems which he has to solve.

* The implement as manufactured and sold has a length of radius of about $4\frac{1}{4}''$, and a circumference for the wheel R of about $2\frac{6}{17}''$, giving 10 as the multiplication. It has been stated in the outset that one complete revolution of this wheel records an area of 10 square inches.

To the Mechanical Engineer it is a great boon, as by its use he is enabled rapidly and with accuracy to find the average pressure upon the piston of a steam-engine as given by indicator diagram: all that is necessary is to ascertain the area of the figure, then to divide that area by the length and the mean height; the representative of the average pressure is at once obtained.

There are, no doubt, other instances in which such an implement is of great use, but the writer feels it is unnecessary to adduce them in support of the claim of the planimeter to the consideration of engineers and of men of cognate professions; and he brings his paper to a conclusion with the expression of a hope that he has by the use of plain, in fact homely, description solved the problem which he set himself in the outset, and has made it clear how it is that the area of any figure, however irregular, can be recorded in definite standard units of measurement by the mere passage of a tracer along the perimeter of that figure.